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Filter Design: Analysis and Review

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Abstract
This paper represents the review analysis of various types of filters design. In this paper we discuss the designing of filters. For discussing the designing of filters we consider the some standard paper which is based on filter design. First we will discuss about filter then we will discuss about types of filter and give the review on different ways of designing of filter.

I.   Introduction

A filter is generally a frequency-selective device. Some frequencies are passed through the filter and some frequencies are blocked by filter. The frequencies of signals that are pass through the filter are called pass band frequencies and those frequencies that are stop by the filter are call stop band frequencies. For pass band frequencies, the system function magnitude is very large and ideally is a constant while for a stop band frequencies, the system function magnitude is very small and ideally is zero. The filter is also work as a remove the noise component from the signal and passes the remaining signal. Ideally a good filter removes the noise component completely from the signal without any loss of meaningful part of signal. Basically, filters are two types:-
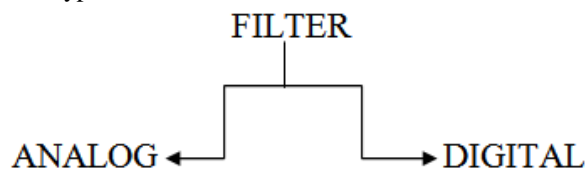

Fig 1.1 Description of filters

Analog filter is simply to implement and requires few electrical components like resistors, capacitors and inductors while for implementation of digital filter, first we convert the ananlog signal into digital signal by taking a samples value of analog signal then we implement the digital filter with the help of adders, subtracts, delays etc. which are classified under digital logic components. Analog filter's characteristics are fixed by circuit design and component values. If we want to change the filter characteristics than we have to make major modification in circuit while in digital filters do not require the major modification. But in digital filter we remove the noise easily from the signal as compare to analog filter that's why generally we use digital filters.
Another classification of filters is:-

- ✓ Butterworth Filter
- ✓ Chebyshev Filter
- ✓ Eliptical Filter
- ✓ Linkwitz-Riley Filter
- ✓ Eletronic Filter
- ✓ Biquad Filter
- ✓ Notch Filter
- ✓ Comb Filter
- ✓ Infinite and Finite Impulse Response Filter (IIR and FIR)
- ✓ High Pass and Low pass Filter (HPF and LPF)
- ✓ Bessel Filter
- ✓ Bilinear Filter
- ✓ Adaptive Filter
- ✓ Kalman Filter
- ✓ Wiener Filter
- ✓ ARMA Filter

In this paper we will take some filters which are mention above and comparison the various type of designing on different filters

II.   BUTTERWORTH FILTER:-

The magnitude- frequency response of low-pass Butterworth filters are characterized by [1]

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega c)^{2N}} = \frac{1}{1+ \in^2 (\Omega/\Omega p)^{2N}}$$

(2.1)

Where A is a filter gain, N is a order of a filter, Ωc is a -3db frequency, Ωp is the pass band edge frequency and $\frac{1}{1+\in^2}$ is the band edge value of the $|H(\Omega)|^2$. In Butterworth filter the magnitude response decreases monotonically as the frequency increases. When the number of order of a filter increases, the Butterworth filter magnitude response more closely approximates the ideal response. The Butterworth filter is also said to have maximally flat

response. The phase response of a butterworth filter becomes nonlinear as N increase.

2.1 Designing Method
Paper 1

Butterworth low pass filter is also used for a remove some noise from the signals like electrical noise, drift rates etc. in navigation system. The navigation system plays an important role to determine the position or attitude in the dynamics or stationary condition. So, **Ping et al** [2] designed an active low pass Butterworth filter using symbolic codes with the help of Sallen key topology structure. For designing a Butterworth low pass filter they used an operational amplifier as the active component like some resistors and capacitors which provide an RLC like filter performance. They take a standard and discrete value of resistors and capacitors then select the cut-off frequency. They used a Sallen-key topology to design a transfer function. Sallen-key circuit is look like a voltage control voltage source. Sallen- key topology is better if
- ✓ Gain accuracy is important
- ✓ Unity gain filter is required
- ✓ Pole-pair Q (quality factor) is low

Ping et al make 2nd order Butterworth low pass filter from Sallen-key topology then for 4th order Butterworth low pass filter they used a cascade connection. After that for every stage MATLAB code used and simulated into PSPICE. This Butterworth filter is the combination of resistors, capacitors and op-amp. It is very reliable and functional filter but this process of designing is very slow. This filter provides better analysis and verification.

Paper 2

Another method of designing a Butterworth low-pass filter is given by **Li Zhongshen** [3]**.** They present the nth-order improved Butterworth low-pass filter, whose n poles are chosen from the poles of (n+2)th-order Butterworth low-pass filter. The conventional Butterworth low-pass filter exist the contradiction problems between test precision, stability and response time. For improvement this type of problem they designed the improved Butterworth low-pass filter.

Butterworth low-pass filter is a type of whole-pole filter, each pole of the Butterworth low-pass filter is distributed on the complex plane. Phase-angle difference of two bordering poles is $180^0 / n$, the angle θ between the pole which is nearest to image axis and image axis is $180^0 / 2n$. For 4th order Butterworth low-pass filter, the angle of θ = 22.5^o and for 5th order θ = 18°. They notice that the conjugate complex poles distribute far from and close with image axis with increasing the order n of the

Butterworth low-pass filter. So the damping factor δ is close to 1 and 0. The conjugate complex poles distribute closer with image axis, the corresponding damping factor δ is smaller, the overshoot is larger, and the system has poor stability.

For improving the stability of nth-order Butterworth low-pass filter they chosen the poles from (n+2)th order of Butterworth low-pass filter. For example, for designing of 4th order Butterworth low-pass filer they choose the poles from 6th order Butterworth low-pass filter. They choose four poles far from image axis as poles of improved Butterworth low-pass filter and the angle θ of 4th order improved Butterworth low-pass filter will equal 45°. Similarly, for designing of 5th order Butterworth low-pass filter they choose the poles from 7th order Butterworth low-pass filter and the angle of θ will equal to 38.57^o.

They observed that if the order n of the improved filter is even, then the transfer function of the improved filter is given as

$$G(\text{s}) = \cfrac{1}{\displaystyle\prod_{k=1}^{\frac{n}{2}}\left(\frac{s^2}{w_c^2} + 2(\cos\theta_k)\frac{s}{w_c} + 1\right)}$$

Where

$$\theta_k = \frac{(2\,\text{k}-1)\pi}{2(\text{n}+2)}$$

If the order n of the improved Butterworth Filter is odd, then the transfer function of the improved filter is given as

$$G(\text{s}) = \cfrac{1}{\left(\dfrac{s}{w_c}+1\right)\displaystyle\prod_{k=1}^{\frac{n-1}{2}}\left(\frac{s^2}{w_c^2} + 2(\cos\theta_k)\frac{s}{w_c} + 1\right)}$$

Where

$$\theta_k = \frac{k\pi}{n+2}$$

Now, the improved Butterworth low-pass filter is analyzed. They compared with the conventional Butterworth low-pass filter f the same order, the improved Butterworth low-pass filter has rapid response, smaller overshoot, quality of unit step response is more ideal and good stability. This kind of filter can be used to enhance the test precision.

Paper 3

When we increase the order of Butterworth low pass filter then the filter will approximate the ideal filter characteristics. But it will increase the cost and circuit complexity of the filter. So **Md Idro and Abu Hasan** [4] discovered the filter's order on the ideal filter approximation due to this cost and circuit complexity reduced. For this they used the Sallen-key

topology, because Sallen-key topology requires fewer components and they assumed that resistors and capacitors are assumed to be equal. They used an operational amplifier to reduce the problem of uniformly and balancing due to sensitive of analog signal circuit variation in electrical parameter. For increasing the order of filter they used cascading of lower order filter. The process is show in the flow chart 2.1. First, they design the basic two stages op-amp. The op-amp is design based on the following specification:-

1. Phase Margin $> 60^0$
2. Unity Gain Frequency > 5MHz
3. DC Gain > 100 (40 dB)
4. Voltage Swings $= 1.0$V to 4.0V
5. Slew rate > 5V/μs

Some equations are related to the above specifications. By implement both specifications parameters and equations they design a basics two stages op-amp and determine the transistor size (W/L).
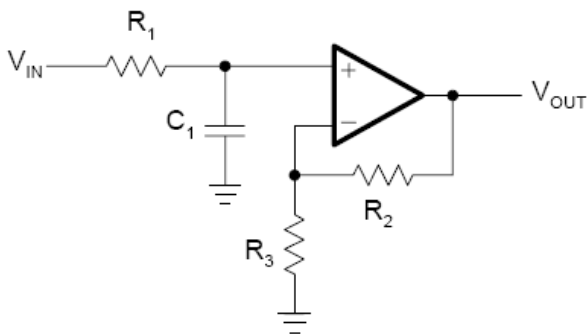
After that they designed a first order LPF.

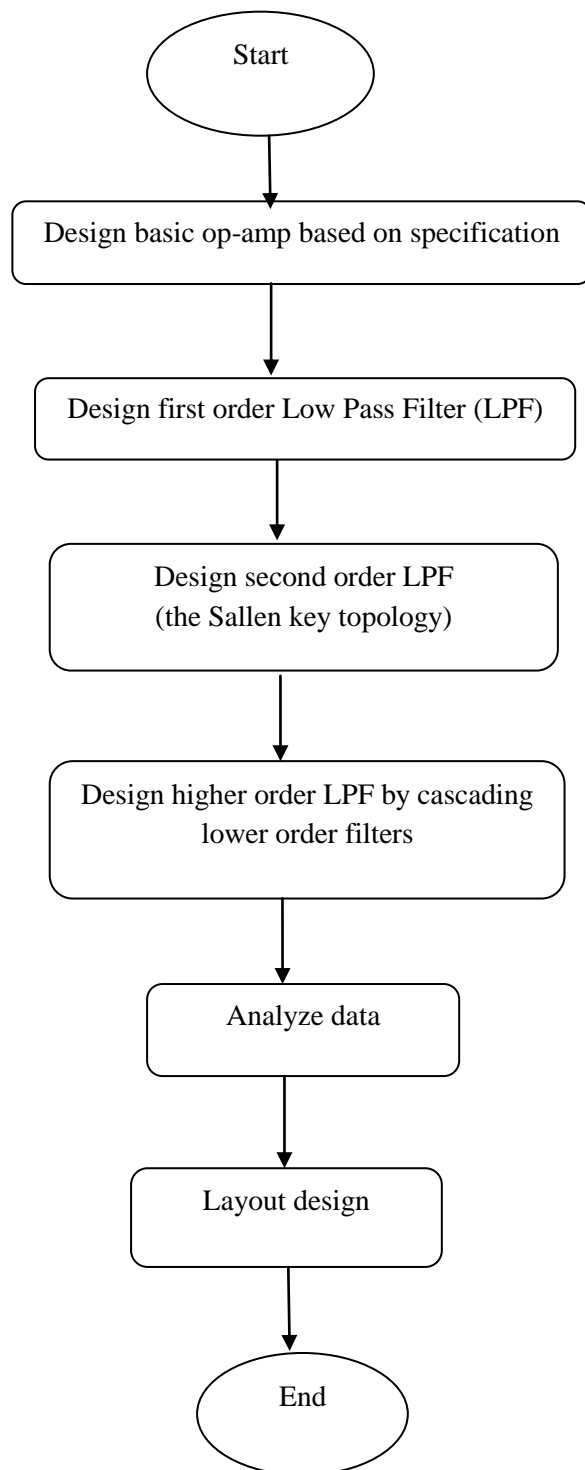**Fig 2.1:- First-order non-inverting LPF**

Fig 2.1:- Flow chart of Process of the Experiment

The transfer function for the above circuit is

$$A(\text{s}) = \frac{1 + \dfrac{R_2}{R_3}}{1 + w_c R_1 C_1 s}$$

Where DC gain $A_0 = 1 + \dfrac{R_2}{R_3}$ and

$$R_1 = \frac{a_1}{2\pi f_c C_1}$$

$$R_2 = R_3(A_0 - 1)$$

Where value of coefficient a_1 is taken from Butterworth coefficient tables and calculate the value of resistor R_1 and R_2. From these values they design the first-order LPF.

After the designing of first-order LPF they designed the second-order LPF using the Sallen-key tpology. The general Sallen-key LPF is given below
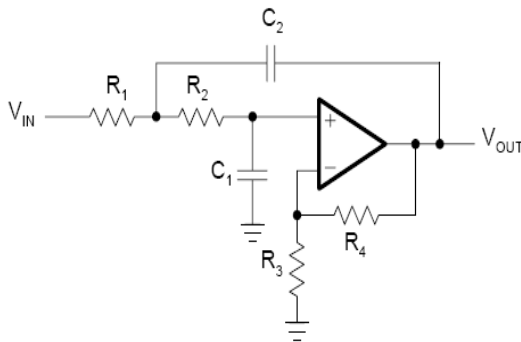

Fig 2.2:- Generall Sallen-key LPF

The transfer function of the above figure 2.2 is:

$$A(\text{s}) = \frac{A_0}{1 + w_c[C_1(R_1 + R_2) + (1 - A_0)R_1 C_2]s + w_c^2 R_1 R_2 C_1 C_2 s^2}$$

Since the Sallen-key equal component is used, the transfer function of figure 2.2 would be different from above equation

$$A(\text{s}) = \frac{A_0}{1 + w_c RC(3 - A_0)s + (w_c RC)^2 s^2}$$

Where

$$a_1 = w_c RC(3 - A_0)$$

$$A_0 = 1 + \frac{R_4}{R_3}$$

$$b_1 = (w_c RC)^2$$

Set the value of C and calculate the R and A_0.

$$R = \frac{\sqrt{b_1}}{2\pi f_c C}$$

$$A_0 = 3 - \frac{a_1}{\sqrt{b_1}} = 3 - \frac{1}{Q}$$

The value of A_0 depends only on quality factor Q and vice versa. After that the final step is design the layout for first order Butterworth LPF. So Sallen-key topology is good choice to reducing the circuit complexity.

Paper 4

Another design is proposed by **Gilbert and Fleming** [5]. They design a high order of Butterworth filter with using either series connection or parallel connection or combination of both connections. They design a filter by using double precision method to overcome the numerical accuracy rather than a direct method. Due to this method the simplicity and economy of programming are easy but the programming become tough compare to direct form.

For designing a Butterworth filter using a direct expansion method they obtained a transfer function which a digital filter may be realized is given in the form of z- transform

$$H(z) = \frac{\displaystyle\sum_{p=0}^{N} E_p z^{-p}}{\displaystyle\sum_{p=0}^{N} F_p z^{-p}} \qquad (2.2)$$

Where $E_0, E_1, \dots E_N$ and $F_0, F_1, \dots F_N$ are the filter coefficients and N is the order of filter.

For the digital Butterworth low-pass filter they applied bilinear transformation in equation (2.2)

$$H(z) = \frac{1}{\displaystyle\sum_{p=0}^{N} a_p \left[\dfrac{2(1 - z^{-1})}{T(1 + z^{-1})} \right]^p} \qquad (2.3)$$

Where

$$a_p = \Omega_c^{-p} \prod_{m=1}^{p} \left\{ \frac{\cos\left[\dfrac{(m-1)\pi}{2N} \right]}{\sin\left(\dfrac{m\pi}{2N} \right)} \right\} \qquad p = 1, 2, \dots, N$$

$$a_0 = 1$$

T is the sampling period, and Ω_c is the cut off frequency in rad/sec. They expand powers of $(1 + z^{-1})$ and $(1 - z^{-1})$ using Binomial expansion

$$H(z) = \frac{\sum\limits_{p=0}^{N} z^{-p} C_p^N}{\sum\limits_{r=0}^{N} \left\{ a_r \left(\frac{2}{T} \right)^r \sum\limits_{p=0}^{r} [(-1)^p z^{-p} C_p^r] \sum\limits_{q=0}^{N-r} [z^{-q} C_q^{N-r}] \right\}}$$

(2.4)

Where

$$C_p^N = \frac{N!}{(N-p)!\,p!} \quad 0 \le p \le N$$
$$= 0 \quad otherwise$$

It should be noted that the summation variables in the denominator of (2.4) are rearranged at this stage in order to arrive at the final form of H(z) with the desired summation variable p. They defined

$$A_p(\mathrm{r}) = (-1)^p C_p^r$$

$$B_q(\mathrm{r}) = C_q^{N-r}$$

They took a denominator of equation (2.4) and rearranged it in the following way:-

$$\sum\limits_{p=0}^{r} \left[(-1)^r z^{-p} C_p^r \right] \sum\limits_{q=0}^{N-r} \left[z^{-q} C_q^{N-r} \right] = \sum\limits_{p=0}^{N} z^{-p} D_p(\mathrm{r})$$

Where

$$D_p(\mathrm{r}) = \sum\limits_{p=0}^{p} A_q(\mathrm{r}) B_{p-q}(\mathrm{r})$$

In the above expansion, terms are included for convenience that may be zero, e.g., $B_N(\mathrm{r}) = 0$ if r > 0.

From the denominator of equation (2.4)

$$\sum\limits_{r=0}^{N} \left\{ a_r (2/T)^r \sum\limits_{p=0}^{N} z^{-p} D_p(\mathrm{r}) \right\} = \sum\limits_{p=0}^{N} \left\{ z^{-p} \sum\limits_{r=0}^{N} a_r \left(\frac{2}{T} \right)^r D_p(\mathrm{r}) \right\}$$

Hence, equation (2.4) can be written as a

$$H(z) = \frac{\sum\limits_{p=0}^{N} z^{-p} C_p^N}{\sum\limits_{p=0}^{N} \left\{ z^{-p} \sum\limits_{r=0}^{N} a_r \left(\frac{2}{T} \right)^r D_p(\mathrm{r}) \right\}}$$

This is the direct form of equation (2.2) with the filter coefficients given by

$$E_p = C_p^N$$

$$F_p = \sum\limits_{r=0}^{N} a_r \left(\frac{2}{T} \right)^r D_p(\mathrm{r})$$

Using double precision method they designed the direct form of Butterworth low-pass filter. They applied the double precision method only in denominator of transfer function and calculated the denominator coefficients, since the numerator coefficients are integer. They observed that due to this method the simplicity and economy of programming are easy but the programming become tough compare to direct form.

2.2 Discussion:-

Table 1 provides an analysis of designing of Butterworth filter. Sallen-key topology provides functional and reliable filter. It is also provide the fast method to analysis and verification of filter. And in another designing of Butterworth filter Sallen-key topology is used to reduce the cost and complexity the filter. But Sallen-key topology has a slow designation process. In double precision method, the simplicity and economy of programming are easy but programming is tough compare to direct form.

2.3 Comparison Table:-

Paper	Approach	Strength	Weakness
Paper 1	Filter design using symbolic codes with help of sallen-key topology	Reliable and functional filter. Fast method to analysis and verification	Design process is slow
Paper 2	Take a nth order pole from (n+2)th order filter to improve the filter	Better frequency performance. Smaller overshoot and more stability	
Paper 3	Increase the number of order with the help of sallen-key topology due to this filter behaves like a ideal filter.	Reduce the cost and complexity	
Paper 4	Design a filter using double precision method to overcome the numerical accuracy rather than direct form	Simplicity and economy of programming are easy	Applied the double precision method only in denominator of transfer function and calculated the denominator coefficients,

			since the nearest to origin maximum of frequencies but not
			numerator modifying in the normalized edge frequency.
			coefficients
			are integer.

Table 2.1:- Comparison Table of butterworth filter desing

III. 3.Chebyshev Filter

There are two types of chebyshev filters [1]. Type 1 chebyshev filters are all pole filters and exhibit equiripple behaviour in the pass band and a monotonic characteristic in stop band. Type 2 chebyshev filters contain both poles and zeros and exhibit a monotonic behaviour in pass band and equiripple behaviour in stop band. The zeros of this type of filter lie on the imaginary axis in the s-plane.

The magnitude squared of the frequency response characteristics of a type 1 chebyshev filter is given as

$$|H(\Omega)|^2 = \frac{1}{1+ \in^2 T_N^2(\frac{\Omega}{\Omega_p})} \qquad (3.1)$$

Where $\in$ is a parameter of the filter related to the ripple in passband and $T_N(x)$ is the Nth order chebyshev polynomial defined as

$$T_N(x) = \cos(N\cos^{-1} x) \qquad |x| \leq 1 \qquad (3.2)$$

$$T_N(x) = \cosh(N\cosh^{-1} x) \qquad |x| > 1 \qquad (3.3)$$

The chebyshev polynomials can be represented by a recursive equation

$$T_{N+1}(x) = 2xT_N(X) - T_{N-1}(x) \quad N = 1,2..... \qquad (3.4)$$

The type2 chebyshev filter magnitude squared of frequency response is given by

$$|H(\Omega)|^2 = \frac{1}{1+ \varepsilon^2 \left[\dfrac{T_N^2\left(\dfrac{\Omega_s}{\Omega_p}\right)}{T_N^2\left(\dfrac{\Omega_s}{\Omega}\right)}\right]} \qquad (3.5)$$

Where $T_N(x)$ is the Nth order chebyshev polynomial and is the stopband frequency.

3.1 Designing Method
Paper 1

In designing and analysis of chebeyshev filter **Lopez and Fernandez** [6] give a chebyshev filter transfer function with improvement of delay response. The technique consists in the shifting of the nearest to origin maximum of frequencies but not modifying in the normalized edge frequency.

Chebyshev filters phase variation depends upon the chebyshev polynomial order, that is, if the chebyshev filter has greatest polynomial order then filter gives the worst phase response. Using this technique Lopez et al give the improvement in the delay response of chebyshev filter. For this technique Lopez et al using equation (3.2) because they use normalized edge frequency $\Omega_c =$1rad/sec. For finding the roots of $T_N(x)$, they have to calculate the value of X such that $T_N(x) =$0 and find the value of x. Now this value put into the chebyshev transfer function and find that no zeros at x=0. Now they modified the chebyshev polynomial such that $T_N(x)$ has a pair of zeros at the origin and edge frequency does not change. This change of variable consists in the shifting of the nearest root to origin at the same origin as two roots equal to zero.

In short we conclude that for low orders filters the delay response will get considerably better after they made the proposed mapping. Additionally to this improvement, the gain response gives us the possibility to make an LC ladder realization t be equally terminated for even order function.

Paper 2

Another method to designing chebyshev filter is given by **Hisham L. Swady** [7]. Swady introduced a new generalized chebyshev filter- like approximation for analog filters. This analyse only for odd order filter. This work presents a general approach for analog filter designing for chebyshev-like approximation with odd order.In this approach chebyshev-like filter differ from classical chebyshev filter in the ripple factor which is not equal amplitude with classical one.

In this method consider the general magnitude squared function of a low pass filter of order N (N=2n+1):-

$$|F(jw)|^2 = \frac{1}{1+\varepsilon^2 P(w^2)} \qquad (3.6)$$

Where

$$\in^2 = \delta_p^{-2} -1$$

Where δ_p is the passband gain and is the minimum value of a |H(Ω)|.This approach is equal to classical chebyshev filter authr satisfy the following condition:

$$\varepsilon^2 P(w^2) << 1 \qquad 0<<w<<wp$$

$$\varepsilon^2 P(w^2) >> 1 \qquad w>>ws$$

Swady approach depends on choosing $\delta_p \leq |H(\Omega)| \leq 1$ in passband and obtained a formula after server manipulations:-

$$\max[P(w^2)] \leq 1 \qquad (3.7)$$

For a special case with choosing $P(w^2) = T_N^2(w)$ equation (3.5) becomes identical to magnitude squared function of the chebyshev filter. The value of n is determined from the stopband requirement that satisfy:

$$|F(jw)| \leq \delta_s \ , \ w \geq w_s \qquad (3.8)$$

Where δ_s and w_s are the stopband tolerance and stopband edge frequency. By an analytical manipulation Equ(3.8) can be transformed into the following result:

$$P(w^2) \geq \frac{\delta_s^{-2} - 1}{\delta_p^{-2} - 1} \qquad (3.9)$$

From the equ (3.7) and (3.9) calculate the filter parameters and pole location of the filter.

From the observation it is clear that the new proposed approximation has non-equal ripple in passband and more flatness than chebyshev filter (as shown in fig) and improvement in time domain response can be observed.
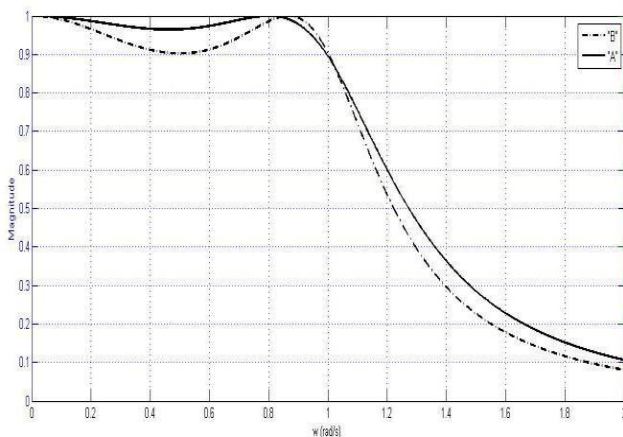
Fig.3.1: Passband performance of the proposed design filter ("A") & the Chebyshev filters ("B")

Paper 3

Another method of design a chebyshev filter is given by **Sunil Bhooshan and vinay kumar** [8]. They proposed a designing a 2-dimensional chebyshev non-recursive filter with linear phase for processing image. In this designing of filter pass band to stop band ratio is critical because they design for the side bands to be a certain amount lower than the pass band.

To convert the 1D Chebyshev polynomial, defined in equation (3.2) and (3.3), to 2D they change the variable X into a new variable ρ, which represent the chebyshev polynomial in cylindrical coordinates system. Where $\rho = \sqrt{x^2 + y^2}$. Now chebyshev polynomial can be represented in the form of new variable ρ a $T_N(\rho) = \cos(N\cos^{-1}\rho)$ -1<|ρ|<1 (3.10)

$$T_N(\rho) = \cos(N\cosh^{-1}\rho) \qquad |\rho|>1 \qquad (3.11)$$

By the use of special transformation ρ can be represented as

$$\rho = \rho_0 \cos(\frac{w}{2}) \quad -\pi \leq w \leq \pi \qquad (3.12)$$

Where ρ_0 is the maximum value of ρ. From the equation (3.12) it is notice that as w varies from 0 to π, ρ will vary from ρ_0 to 0 and vice versa. Therefore, they conclude that this transform converts the polynomial to low pass filter. if we wish to design a high pass filter from the polynomial than ρ can be represented as:

$$\rho = \rho_0 \sin(\frac{w}{2}) \quad -\pi \leq w \leq \pi \qquad (3.13)$$

Now from equation (3.12) for low pass filter polynomial can be defined as:

$$T_N(\rho) = \cos\{N\cos^{-1}\rho_0(\frac{\sqrt{u^2+v^2}}{2})\} \quad -1<|\rho|<1 \ (3.14)$$

$$T_N(\rho) = \cos\{N\cosh^{-1}\rho_0(\frac{\sqrt{u^2+v^2}}{2})\} \quad |\rho|>1 \qquad (3.15)$$

Where w= $\sqrt{u^2+v^2}$ is mapped onto a 2D plane of frequencies. Similarly, we can represent the polynomial for high pass filter. To realize this filter ρ_0 can be represented in term of order of filter and attenuation require for side bands.

$$\rho_0 = \cosh(\cosh^{-1}b / N) \qquad (3.16)$$

Where N is the order of the filter and b is the absolute value of the attenuation and is defined by

$$b = 10^{(attenuation \ in \ dB/20)} \qquad (3.17)$$

They also defined the value of stop band frequency w_s and pass band frequency w_p as

$$w_s = 2\cos^{-1}[1/\{\cosh(\overline{1/N}\cosh^{-1}b)\}] \qquad (3.18)$$

$$w_p = 2\cos^{-1}\left[\frac{\cosh\{(1/N)\cosh^{-1}(b/\sqrt{2})\}}{\cosh(\overline{1/N}\cosh^{-1}b)}\right]$$
$$(3.19)$$

Now from this approach we can design a 2D linear filter with the help of chebyshev polynomial. They conclude that if the order of the filter increases than the transition band becomes sharper and sharper. In this approach no approximation involved,

computation time is less and it is easier to understand.

3.2 Discussion

In the first paper **Lopez and Fernandez** give the desigining of modified chebyshev filter by shifting the frequency near to origin. Due to this improvement in the delay response of the chebyshev filter but it is only for even order chebyshev filter. After that we discuss the paper of **Hisham L. Swady** where they give the designing of chebyshev filter by using analog filter. They used the magnitude response of low pass filter of order N and convert into chebyshev filter. Due to this technique we get the more flat response in the pass band frequency range but this technique is valid only for odd order chebyshev filters. After that we discuss the paper of **Sunil Bhooshan and Vinay Kumar** where they give the designing of a 2D chebyshev filter using the 1D chebyshev polynomials. It is very easy to design, no approximation involved and complexity is less.

3.3 Comparison Table

Paper	Approach	Strength	Weakness
Paper 1	Shifting the frequencies nearest to origin and not change in the cut-off frequency	Improvement in delay response	Only for even order chebyshev filter
Paper 2	use magnitude response of low pass filter and convert into chebyshev filter magnitude response	Mangnitude response is more flat in pass band frequencies	Only for odd order chebyshev filter
Paper 3	Convert 1D chebyshev polynomial int 2D polynomial	Complexity is less and no approximation involved	

Table 3.1:- Comparison Table of chebyshev filter desing

IV.   Notch Filter

Notch filter [1] contains one or more deep notches. It has perfect nulls in its frequency response characteristics. Notch filters are very useful in many applications where the specific frequency components must be eliminated. For example, instrumentation and recording systems require that the power line frequency 60 Hz and its harmonics be eliminated. For creating a null in the frequency response of a filter at a frequency w_0, we simply introduce a pair of complex-conjugate zeros on the unit circle at an angle w_0. That is,

$$z_{1,2} = e^{\pm jw_0}$$
(4.1)

Thus the system function for an FIR notch filter is

$$H(z) = b_0(1 - e^{iw_0}z^{-1})(1 - e^{-iw_0}z^{-1})$$
$$= b_0(1 - 2\cos w_0 z^{-1} + z^{-2})$$
(4.2)

But in the FIR notch filter notch has a relatively large bandwidth, which means that other frequency components around the desired null are severely attenuated. To reduce the bandwidth of the null we introduce poles in the system function. Suppose that we replace a pair of complex-conjugate poles at

$$p_{1,2} = re^{\pm jw_0}$$
(4.3)

The effect of poles is to introduce a resonance in the vicinity of the null and thus to reduce the bandwidth of the notch. The system function for the resulting filter is

$$H(z) = b_0 \frac{1 - 2\cos w_0 z^{-1} + z^{-2}}{1 - 2r\cos w_0 z^{-1} + r^2 z^{-2}}$$
(4.4)

In addition to reducing the bandwidth of the notch, the introducing of a pole in the vicinity of the null may result in a small ripple in a pass band of the filter due to the resonance created by poles. The effect of ripple can be reduced by introducing poles and/or zeros in the system function of notch filter.

4.1 Designing Method
Paper 1

In designing and analysis of Notch filter **Lee and Tseng** [9] give the design of 2D notch filter using band pass filter and fractional delay filter. For designing of 2D notch filter decompose the filter into the 2D parallel-line filter and straight-line filter. Then, the parallel-line filter is designed by band pass filter and the straight-line filter is designed by fractional delay filter. The purpose of this paper is to establish the relation between 2D notch filter and the fractional delay filter such that 2D notch filter can be designed by using well-documented design method of fractional delay filter.

The ideal frequency response of 2D notch filter is given by

$$D(w_1, w_2) = 0 \ , \ (w_1, w_2) = \pm(w_{1N}, w_{2N})$$
$$= 1, others$$
(4.5)

Where (w_{1N}, w_{2N}) is the notch frequencies. Now design a 2D filter $H(z_1, z_2)$ which is approximate the $D(w_1, w_2)$. They choose the simple algebraic method to make the IIR notch filter. In the simple algebraic method, the design of 2D notch filter $H(z_1, z_2)$ is decompose into 2D parallel-line filter $H_p(z_1, z_2)$ and 2D straight-line filter $H_s(z_1, z_2)$. The transfer function of 2D notch filter is given by

$$H(z_1, z_2) = 1 - H_p(z_1, z_2)H_s(z_1, z_2)$$
(4.6)

Where the parallel-line filter is

$$H_p(z_1, z_2) = \frac{1}{2}\left(1 - \frac{a_2 - a_1 z_2^{-1} + z_2^{-2}}{1 - a_1 z_2^{-1} + a_2 z_2^{-2}}\right)$$
(4.7)

With the coefficients

$$a_1 = \frac{2\cos(w_{2N})}{1 + \tan\left(\frac{BW}{2}\right)}, \quad a_2 = \frac{1 - \tan\left(\frac{BW}{2}\right)}{1 + \tan\left(\frac{BW}{2}\right)}$$

BW is a small positive number. Now for designing straight-line filter they used the analog inductance-resistance network and bilinear transformed.

$$H_s(z_1, z_2) = \frac{1 + z_1^{-1} + z_2^{-1} + z_1^{-1}z_2^{-1}}{\frac{R + L_1 + L_2}{R} + \frac{R - L_1 + L_2}{R}z_1^{-1} + \frac{R + L_1 - L_2}{R}z_2^{-1} + \frac{R - L_1 - L_2}{R}z_1^{-1}z_2^{-1}}$$
(4.8)

Where the parameters are

$$L_1 = \frac{1}{\tan\left(\frac{w_{1N}}{2}\right)}, \quad L_2 = \frac{-1}{\tan\left(\frac{w_{2N}}{2}\right)}$$

and R is a small positive number. For the bounded input/bounded output (BIBO) stability of the 2D notch filter, the parameters need to satisfy the constraints $L_1 > 0$ and $L_2 > 0$. This implies that notch frequency needs to satisfy the following condition:-

$$w_{1N} > 0 \text{ and } w_{2N} < 0$$

The above constraints limit the applicability of notch filter. so they used the fractional delay filter to design the straight-line filter such that constraint can be removed.

Now the proposed design method divides into 3 following case:

Case-1: when $|w_{1N}| > |w_{2N}|$, then parallel-line filter is design by the 1D band pass filter.

$$H_p(z_1, z_2) = H_{B1}(z)|_{z=z_2}$$

Where H_{B1} is the 1D band pass filter. Moreover, the straight-line filter is given by

$$H_s(z_1, z_2) = H_{B2}(z)|_{z=z_1^{\alpha}z_2}$$

Where H_{B2} is another 1D band pass filter and $\alpha = -\frac{w_{2N}}{w_{1N}}$. Now put the value in the equation (4.6), the design notch filter is given by

$$H(z_1, z_2) = 1 - \left(1 - \frac{1 + az_2^{-1} + z_2^{-2}}{1 + raz_2^{-1} + r^2 z_2^{-2}}\right)\left(1 - \frac{1 - z_1^{-\alpha}z_2^{-1}}{1 - rz_1^{-\alpha}z_2^{-1}}\right)$$
(4.9)

Where r is the small positive number and $a = -2\cos(w_{2N})$.

Case 2: When $|w_{1N}| < |w_{2N}|$, then parallel-line filter is design by the 1D band pass filter.

$$H_p(z_1, z_2) = H_{B1}(z)|_{z=z_1}$$

Where H_{B1} is the 1D band pass filter. Moreover, the straight-line filter is given by

$$H_s(z_1, z_2) = H_{B2}(z)|_{z=z_1 z_2^{\beta}}$$

Where H_{B2} is another 1D band pass filter and $\beta = -\frac{w_{1N}}{w_{2N}}$. Now put the value in the equation (4.6), the design notch filter is given by

$$H(z_1, z_2) = 1 - \left(1 - \frac{1 + az_1^{-1} + z_1^{-2}}{1 + raz_1^{-1} + r^2 z_1^{-2}}\right)\left(1 - \frac{1 - z_1^{-1}z_2^{-\beta}}{1 - rz_1^{-1}z_2^{-\beta}}\right)$$
(4.10)

Where $a = \cos(w_{2N})$.

Case 3: if $w_{1N} = w_{2N}$, then 2D notch filter is

$$H(z_1, z_2) = 1 - \left(1 - \frac{1 + az_1^{-1} + z_1^{-2}}{1 + raz_1^{-1} + r^2 z_1^{-2}}\right)\left(1 - \frac{1 - z_1 z_2^{-1}}{1 - rz_1 z_2^{-1}}\right)$$
(4.11)

And if $w_{1N} = -w_{2N}$, then 2D notch filter is

$$H(z_1, z_2) = 1 - \left(1 - \frac{1 + az_1^{-1} + z_1^{-2}}{1 + raz_1^{-1} + r^2 z_1^{-2}}\right)\left(1 - \frac{1 - z_1^{-1}z_2^{-1}}{1 - rz_1^{-1}z_2^{-1}}\right)$$
(4.12)

Where $a = \cos(w_{2N})$. This is designing of the 2D notch filter using band pass filter and fractional delay filter. This is the close form design, so it is easy to use.

Paper 2

Another method of designing of notch filter is given by **Tseng and Pei** [10]**.** In this paper they represent the designing of 2D IIR Notch filter and 2D FIR Notch filter. They explain the 2D IIR Notch filter by simple algebraic method which we have discussed in the above paper and 2D FIR Notch filter explain by Lagrange method. FIR filter requires more arithmetic operation than IIR filter during implementation.

A 2D linear phase FIR digital filter with transfer function $H(z_1, z_2)$ has a frequency response

$$H(e^{iw_1}, e^{jw_2}) = e^{-jN_1 w_1} e^{-jN_2 w_2} \sum_{m=-N_1}^{N_2} \sum_{n=-N_1}^{N_2} h(m,n) e^{-j(mw_1 + nw_2)}$$

(4.13)

Where the impulse response h(m,n) satisfy the half-plane symmetrical condition, i.e.

h(m,n) = h(-m,-n)   (4.14)

After some manipulation it is easily shown that the magnitude response $G(w_1, w_2)$ of $H(z_1, z_2)$ is given by

$$G(w_1, w_2) = \sum_{i=1}^{R} a(i) g(w_1, w_2)$$

(4.15)

Where the dimension

$R = (N_1 + 1) + N_2(2N_1 + 1)$ and

$g_i(m,n) = \cos(mw_1 + nw_2)$ (4.16)

$$a(i) = h(0,0) \quad i = 1$$
$$= 2h(m,n) \quad i \neq 1$$

(4.17)

They rewrite the $G(w_1, w_2)$ in the following vector form

$$G(w_1, w_2) = A^t C(w_1, w_2)$$

(4.18)

Where

$$A = [a(1) \quad a(2)....a(R)]^t$$
$$C(w_1, w_2) = [g_1(w_1, w_2) \quad g_2(w_1, w_2)....g_R(w_1, w_2)]^t$$

They design $G(w_1, w_2)$ to approximate the ideal notch filter frequency response in least square error sense. In this sense, the filter design problem formulated as follow:

$$Minimize \quad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |G(w_1, w_2) - 1|^2 \, dw_1 dw_2$$

$$= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |A^t C(w_1, w_2) - 1|^2 \, dw_1 dw_2$$

$$= A^t Q A - 2P^t A + 1$$

$$Subjected \quad to \quad C^t(w_{1N}, w_{2N}) A = 0$$

Where the positive definite symmetric matrix Q and vector P are given by

$$Q = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} C(w_1, w_2) C^t(w_1, w_2) \, dw_1 \, dw_2$$

$$P = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} C(w_1, w_2) \, dw_1 \, dw_2$$

Therefore, the optimal solution is given by

$$A_{opt} = [1 - Q^{-1} C(w_{1N}, w_{2N}) \{ C^t(w_{1N}, w_{2N}) Q^{-1} C(w_{1N}, w_{2N}) \}^{-1} C^t(w_{1N}, w_{2N})] Q^{-1} P$$

(4.19)

Based on this discussion, they design a 2D FIR Notch filter with notch frequency (w_{1N}, w_{2N}). They used this filter to remove the single sinusoidal interference superimposed on an image. For real-time processing purpose, the 2D notch filter is preferred. In 2D notch filter case, the notch frequency can be chosen exactly the same as the sinusoidal frequency, so there is nearly no information loss and notch filter technique is much more efficient than conventional FFT method in computational complexity.

Paper 3

Another method of designing of Notch filter is given by **Shen et al** [11]**.** In this paper based upon transfer function, they design a digital notch filter to eliminate the 50 Hz noise. The filter was realized by the software implementation in VC++ environment and simulated by MATLAB. After this process power-line interference removes effectively. Its operation was simple and applicable.

According to given sampling frequency, the transfer function could be constructed as:

$$H(z) = \frac{1 - z^{-f_s}}{1 - z^{-\frac{f_s}{50}}}$$

(4.20)

Substituting $z = e^{jw}$ in equation (4.20)

$$H(e^{jw}) = e^{-jw\left(\frac{f_s}{2} - \frac{f_s}{100}\right)} \frac{\sin \frac{w f_s}{2}}{\sin \frac{w f_s}{100}}$$

(4.21)

In the digital domain the range of the frequencies is $0 \leq w < 2\pi$ and in analogue domain $0 \leq f < f_s$, where f_s is the sampling frequency.
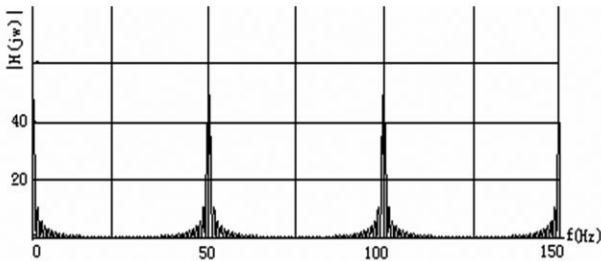
If f= f_s then $w = 2\pi$, and $|H(e^{jw})|$ could simply express as:

$$|H(e^{jw})| = \left| \frac{\sin(\pi f)}{\sin \frac{\pi f}{50}} \right|$$

And the phase shift $\phi = \dfrac{f_s}{2} - \dfrac{f_s}{100}$

$$|H(\text{jw})| = \begin{cases} \approx 0 & f \neq 50.k \quad (k=0,1,2...) \\ \approx 50 & f = 50.k \quad (k=0,1,2...) \end{cases}$$

The magnitude frequency characteristics shown in figure below, which was drawn in MATLAB.

Fig 4.1:- Magnitude Frequency Characteristics of filter

While passing through the transfer function, the series of signals could be demodulated with the frequency component. Notch filter could be obtained by taking the demodulated series from original series. According to transfer function, the differential equation would be calculated.
Assuming that the original signal series was x(n) and demodulated signal series was y(n)

$$y(\text{n}) = \text{y}\left(n - \frac{f_s}{50}\right) + x(\text{n}) - \text{x}(\text{n} - \text{f}_s)$$

Then the band-stopped series $R_e(\text{n})$ could be calculated as follow:

$$R_e(\text{n}) = x\left[n - \left(\frac{f_s}{2} - \frac{f_s}{100}\right)\right] + x(\text{n}) - \text{x}(\text{n} - \text{f}_s)$$

$$\phi = \frac{f_s}{2} - \frac{f_s}{100}$$

The power-line frequency interference has been removed in series $R_e(\text{n})$.

According to the principle of transfer function, the FIR notch filter was designed via software procedure in VC++. Through the specified x(n) and the differential equation f the notch filter, the demodulated series y(n) and $R_e(\text{n})$ could be calculated.In this paper, the transfer function H(z) of digital filter can demodulated the frequency components that was integral multiple of 50 Hz. The experiment showed that the 50 Hz notch filter worked properly with stable and reliable performance.

Paper 4

Another design is presented by the **Srisangngam et al** [12]. This paper represents the design of symmetrical IIR Notch filter using pole position displacement. It has modified the transfer function equation with the optimum pole positions for the symmetry of pass-band gain and transition-band gain. First, they present the IIR Notch filter design then they present IIR Notch filter proposed design and finally simulates the result.
Now the transfer of the IIR Notch filter is given by

$$H(\text{z}) = \text{b}_0 \frac{1 - 2\cos w_0 z^{-1} + z^{-2}}{1 - 2r\cos w_0 z^{-1} + r^2 z^{-2}} \qquad (4.22)$$

From this equation it is show that the angle of pole is the same as angle of zero. The simulation result of this method reveals the uncontrollable gain at frequency DC and π. The constant for controlling the pass-band gain rate b_0 is unknown because of this unknown constant asymmetric and uncontrollable pass-band gain at DC and π frequencies. Then they modified the pole angle and the result will be a transfer function of the IIR Notch filter as shown

$$H(\text{z}) = \text{b}_0 \frac{1 - 2\cos w_0 z^{-1} + z^{-2}}{1 - 2r\cos(w_0 - \phi)z^{-1} + r^2 z^{-2}} \quad (4.23)$$

$$b_0 = \frac{1 - 2r\cos(\text{w}_0 - \phi) + \text{r}^2}{2 - 2\cos w_0} \qquad (4.24)$$

$$\phi = w_0 - \cos^{-1}\left(\frac{1 + r^2}{2r}\cos w_0\right) \qquad (4.25)$$

After changing the pole position of an IIR Notch filter, it could control the magnitude response at DC frequency and π frequency but it could not control transition-band gain. Therefore, the magnitude response of an IIR Notch filter is not symmetry.
Now they add pole and zero to compensate magnitude of transition-band as shown in equation below

$$\hat{H}(\text{z}) = \hat{\text{b}}_0 \, H(\text{z}) \, \text{H}_{ADD}(\text{z})$$

$$\text{H(z)} = \frac{1 - 2\cos w_0 z^{-1} + z^{-2}}{1 - 2r_1\cos(\text{w}_0 - \phi_1)\,\text{z}^{-1} + r_3^2 z^{-2}}$$

$$H_{ADD}(\text{z}) = \frac{1 - 2r_2\cos(\text{w}_0 + \phi_2)\,\text{z}^{-1} + r_2^2 z^{-2}}{1 - 2r_3\cos(\text{w}_0 + \phi_3)\,\text{z}^{-1} + r_3^2 z^{-2}}$$

Because of the zero position has no effect on the transition-band gain and pass-band gain. Thus, ϕ_2 should be assigned as zero and $r = r_1 = r_2 = r_3$.
The $\phi = \phi_1 = \phi_3$ is an angle of pole that will symmetrical with notch frequency in both positive and negative sides.

Where

$$\hat{b}_0 = \frac{1 - 2r\cos(w_0 + \phi) + r^2}{2 - 2\cos w_0} \frac{1 - 2r\cos(w_0 - \phi) + r^2}{1 - 2r\cos w_0 + r^2}$$

The magnitude response of propose IIR Notch filter could control the pass-band gain and transition-band gain. Because of distance from zero position of each side are equal. Then, the magnitude response of an IIR Notch filter is symmetry. In conclusion, the proposed method for the design of IIR Notch filter by modifying the pole position can ensure the symmetry of pass-band and transition-band gain at the target level.

4.2 Discussion:-

In paper 1 **Lee and Tseng** give the design of 2D notch filter using band pass filter and fractional delay filter and using simple algebraic method they design a IIR Notch filter. Due to this technique they remove the sinusoidal interferences corrupted on a desired signal and it is closed-form design, so it is easy to use but they explain only for 2D Notch filter not discuss the 3D notch filter design using same method. In paper 2 **Tseng and Pei** presents the designing of 2D IIR Notch filter and 2D FIR Notch filter. They explain the 2D IIR Notch filter by simple algebraic method 2D FIR Notch filter explains by Lagrange method. We observe that FIR filter requires more arithmetic operation than IIR filter during implementation and they also could not explain the 3D notch filter design using same method. In paper 3 **Shen et al** based upon transfer function design a digital notch filter to eliminate the 50 Hz noise. The filter was realized by the software implementation in VC++ environment and simulated by MATLAB. After this process power-line interference removes effectively. Its operation was simple and applicable. In paper 4 **Srisangngam et al** represents the design of symmetrical IIR Notch filter using pole position displacement. Magnitude response of an IIR Notch filter is symmetry and by modifying the pole position can ensure the symmetry of pass-band and transition-band gain at the target level.

4.3 Comparison Table:-

Paper	Approach	Strength	Weakness
Paper 1	Using simple algebraic method design a IIR Notch filter	Remove the sinusoidal interferences corrupted on a desired signal and it is closed-form design, so it	only fixed 2D notch filter not discuss the 3D notch filter design using same method
		is easy to use.	
Paper 2	2D FIR Notch filter explain by Lagrange method.	For real-time processing purpose, the 2D notch filter is preferred and no information loss and notch filter technique is much more efficient than convention al FFT method in computatio nal complexity.	FIR filter requires more arithmetic operation than IIR filter during implementati on and only fixed 2D notch filter not discuss the 3D notch filter design using same method
Paper 3	Based upon transfer function design a digital notch filter to eliminate the 50 Hz noise with the help of software implement ation in VC++ environme nt and simulated by MATLAB	Power-line interference removes effectively. Its operation was simple and applicable.	
Paper 4	Design of symmetric al IIR Notch filter using pole position displacem ent	Magnitude response of an IIR Notch filter is symmetry and by modifying the pole position can ensure the symmetry	

		of pass-band and transition-band gain at the target level.	

Table 4.1: Comparison table of Notch Filter Design

V.   COMB Filter

Comb filter [1] is a type of notch filter in which nulls occurs periodically across the frequency band, hence comb has periodically space teeth. Comb filters are use in a wide range of practical systems such as in the rejection of power-line harmonics, in separation of solar and lunar components from ionospheric measurements of electron concentration.

To illustrate a simple form of a comb filter, consider a moving average (FIR) filter describe by the difference equation

$$y(n) = \frac{1}{M+1} \sum_{k=0}^{M} x(n-k)$$

The system function of the FIR filter is

$$H(z) = \frac{1}{M+1} \sum_{k=0}^{M} z^{-k}$$

And its frequency response is

$$H(w) = \frac{e^{-jwM/2}}{M+1} \frac{\sin w\left(\frac{M+1}{2}\right)}{\sin\left(\frac{w}{2}\right)}$$

From the above equation we observe that the filter has zeros on the unit circle at

$$z = e^{j2\pi k/M+1}, \quad k = 1, 2, 3, \ldots, M$$

Note that the pole at z=1 is actually cancelled by the zero at z=1, so that in effect the FIR filter does not contain poles outside z=0 [1].

5.1 Designing Method
Paper 1

For designing of Comb filter **Makarov and Odda** [13] present a paper. In this paper the second order COMB filter is consist of two conventional COMB filter in cascade. The purpose of this paper is to simulate this type of filter by using delta modulation (DM) technique which gives the realization low cost, simplicity and an efficient result in real time processing. After that the suggested structure has been simulated through an appropriate computer program to achieve the desired frequency response. This structure is based on DM as analog to digital converter (ADC) for continuous input signal.

The second order COMB filter is:

$$y_1(nT) = x(nT) - 2x(nT-NT) + x(nT-2NT)$$

$$(5.1)$$

Where, x is the discrete sequence obtained from the input signal sampled at time (nT), T is the sampling interval, y is the output sequence and (NT) is the time interval of the delay.   Taking the z-transform of equation (5.1), then we get

$$H_1(z) = \frac{Y_1(z)}{X(z)} = 1 - 2z^{-N} + z^{-2N} = (1-z^{-N})^2$$

$$(5.2)$$

This transfer function has (2N) zeros. Each double zeros are equally spaced around the unit circle in the z-plane at location given by:

$$z_k = e^{j2\pi k/N}, \quad k = 0, 1, 2, \ldots(N-1)$$

The frequency response of the filter is:

$$H_1(e^{-jwT}) = (1-e^{-jwNT})^2$$

Thus,

$$|H_1(e^{-jwT})| = 2(1-\cos(wNT))$$

Which means that zeros occur at the frequencies that are integral multiple of (f_s/N) Hz and f_s is the sampling frequency.

Now, if in equation (5.2), the subtraction is replaced by addition, the transfer function becomes

$$H_2(z) = \frac{Y_1(z)}{X(z)} = 1 + 2z^{-N} + z^{-2N} = (1+z^{-N})^2$$

$$(5.3)$$

Thus, there are double zeros equally spaced around unit circle at

$$z_k = e^{j2\pi(k+0.5)/N}, \quad k = 0, 1, 2, \ldots(N-1)$$

And the frequency response becomes

$$|H_2(e^{-jwT})| = 2(1+\cos(wNT))$$

In this case zeros occur at the frequencies that are integral multiple of $(f_s/2N)$ Hz.

Now, they represent the conventional COMB filter through cascade both transfer function and together with DM are cascade with conventional filter. This is the suggested structure given by them. DM is cascaded with conventional filter's structure in order to convert the analog input signal into uni-bit digital waveform.

Now, the simulated model for second order COMB filter computer program is operated by reading the parameters of the filter and DM. The Mean Squared Error (MSE) is calculated to investigate the performance of suggested structure according to different values of hysteresis width of DM and timing factor of program processing. MSE is defined as follow:

$$MSE = \left(\frac{2}{f_s}\right) \sum_{f=1}^{f_s/2} (H_d(f) - H_s(f))^2$$

Where $H_d(f)$ and $H_s(f)$ are the desired and simulated frequency response respectively.

In this paper, they constructed and designed of second order filter using computer simulation. This design simplifies the structure of this filter using DM as analog to digital converter for input signal. The simulation results are obtained that MSE decrease by increasing clock-frequency and decreasing hysteresis width of DM.

Paper 2

Another method of designing of COMB filter is given by **J. L. Rasmussen** [14]. One application of the COMB filter is to remove of the fundamental sinusoidal signal and its harmonics from a signal of interest (SOI). In this condition, it is not feasible to use simple IIR or FIR filter to remove these interfering signals without significant degradation of SOI. In this case, we can use the FFT COMB filter to remove these interfering signals but the FFT method has not often used because of its computational complexity. In this paper they present a new formulation of the FFT COMB filter that eliminates all limitation of the FFT method and provides advantages over the IIR or FIR COMB filter.

In earlier method of FFT COMB filter collect the N samples of data, performing an FFT on this data, zeroing every M^{th} value of the resulting data, and then performing IFFT to obtain the time domain filtered result. The FFT and IFFT both take order $N \log_2 N$ multiplications or $2N \log_2 N$ multiplications for whole process. Due to this process computational complexity increases. There is another problem in FFT COMB filter if we want to extend this method to larger bandwidth. First, we assume that the original FFT frequency resolution must be maintained. Under this constraint, if the SOI bandwidth is increased, both sampling frequency f_s and N need to be increased. Thus, the FFT size has increased. The FFT/IFFT processing time is also increased.

For solving the above problem of FFT COMB filter they give the new design of FFT COMB filter. They use the FFT and IFFT matrix multiplication process to simplify the above problems. It observe that if N and M are choose such that N/M is an integer, the product of the IFFT matrix with the FFT matrix with every M^{th} rows zeroed produces a sparse, multi-diagonal structure. Let A and A^{-1} be the size N FFT and IFFT matrices. A is given by

$$\frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & \cdots & W_N^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Where $W_N = e^{-j\frac{2\pi}{N}}$

Zeroing every M^{th} value of the result of multiplying the FFT matrix times the input, x, is equivalent to multiplying the input vector by a multiplication of the FFT matrix with every M^{th} row zeroed. Let A_r be the FFT matrix with every M^{th} row zeroed. Thus, A_r is given by

$$\frac{1}{\sqrt{N}} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & W_N^1 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Let C_r be the result of multiplication,

$$C_r = A^{-1} A_r \tag{5.4}$$

A_r can also be represented as the matrix (A-A_e), where A_e is the matrix A with all its rows zeroed out except the first row and every M^{th} row. Then from equation (5.4)

$$C_r = A^{-1}(A - A_e) \quad = I - A^{-1} A_e \tag{5.5}$$

Since only every M^{th} element of the columns of A_e is non-zero, the summation is only over 0 to (N/M-1). The elements of matrix $A^{-1} A_e$ are sums of N/M products and can be expressed as

$$A^{-1} A_e = \begin{cases} \dfrac{1}{M} & n = \mathrm{mod}(i-j, \dfrac{N}{M}) = 0 \\ 0 & otherwise \end{cases}$$

Then, the ij^{th} element of

$$C_r \text{ is}$$

$$C_r(ij) = \begin{cases} 1 - \dfrac{1}{M} & i = j \\ -\dfrac{1}{M} & n = \mathrm{mod}(i-j, \dfrac{N}{M}) = 0, and \quad i \neq j \\ 0 & otherwise \end{cases}$$

$$\tag{5.6}$$

Thus, a multi-diagonal matrix results with 2M-1 diagonal, include the main diagonal. If the input vector, x, is multiplied by this matrix, then in general the result for the kth output is

$$\hat{x}_k = x_k - \frac{1}{M}\sum_{j=0}^{M-1} x_{k-j\frac{N}{M}} \qquad (5.7)$$

For k equal 0 to N/M-1. Because of the cyclic nature of the matrix C_r, only the first N/M outputs need to be calculated as in equation (5.7). The rest of the (N-N/M) outputs can be calculated using the first N/M outputs as

$$\hat{x}_k = \hat{x}_{\text{mod}(k,\,N/M)} + x_k - x_{\text{mod}(k,\,N/M)} \qquad (5.8)$$

For k equal N/M to N.

From this method the total number of multiplications reduces to N/M. Thus, this method of implementation COMB filtering is a factor of $2M\log_2 N$ more efficient than the FFT method. This simplified method solves some of problems that were identified. First, computation time has been reduced. Second, there is no restriction that a radix 2 value for the size of the FFT must be used. The only restriction is that N/M must be an integer. There is still requirement that all N samples must be collected before the output can be obtained. The time involved might be too large for some Time Division Multiple Access or network systems. It can be implemented as an FIR filter. Now, its computational complexity is reduce so we can use this filter is to remove of the fundamental sinusoidal signal and its harmonics from a SOI.

Paper 3

Another method of designing a COMB filter is given by **Gordana Jovanovic Dolecek** [15]**.** This paper presents the design of the 2 stage COMB based decimation filter with a very low wideband pass band droop and a high stop band attenuation of the overall filter. In this paper they used polyphase decomposition method to stage 1 to avoid the filtering at the high input rate. Then, they applied the compensation filters for both stages and sharpening technique is applied in the 2 stage to the cascade of comb filter and the compensation filter.

The comb filter must have a high alias rejection around the zeros of comb filter and a low pass band drop in the pass band in order to avoid the distortion of the decimated signal. However the comb filter has a high pass band droop and low folding band attenuation. So the main goal of this paper is to decrease the pass band droop and keeping good stop band attenuation. In this paper first, they describe two stage structures, then introduce the compensation filter and finally give the proposed filter.

For two stage structure they consider the case where the decimation factor M can be presented as $M = M_1 M_2$. The corresponding comb-based decimation filter is given as

$$H(z) = [H_1(z)]^{k_1}[H_2(z^{M_1})]^{k_2}$$
(5.9)

Where

$$H_1(z) = \frac{1}{M_1}\frac{1-z^{-M_1}}{1-z^{-1}};$$

$$H_2(z^{M_1}) = \frac{1}{M_2}\frac{1-z^{-M_1 M_2}}{1-z^{-M_1}}$$

(5.10)

The choice of M_1 is a matter of the compromise between having less complex polyphase components in the first stage, and making the filter in the second stage working at, as much as possible lower rate. For this condition they proposed that the factors M_1 and M_2 are close in values as much as possible to each other, with $M_1 \leq M_2$. k_1 and k_2 are the number of cascade comb filters at first and second sections respectively. The filter in the first section is realized in the non recursive form,

$$H_1^{k_1}(z) = \left[\frac{1}{M_1}\sum_{i=0}^{M_1-1} z^{-i}\right]^{k_1} \qquad (5.11)$$

Using ployphase decomposition, the filters of the first section can be moved t operate at a lower rate.Two-stage structure is shown in figure
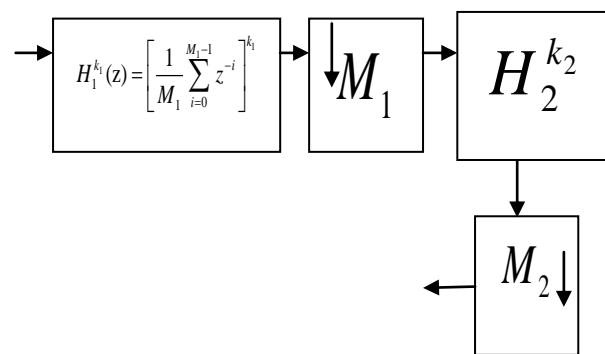

Fig 5.1:- Two-Stage Structure

Now, they try to improve the pass band characteristics of the overall structure. To improve the overall pass band characteristics, they propose to use compensation filters:

$$G_1(z^{M_1}) = [B(1 + Az^{-M_1} + z^{-2M_1})]^{k_{11}} \qquad (5.12)$$

$$G_2(z^M) = [B(1 + Az^{-M} + z^{-2M})]^{k_{22}} \qquad (5.13)$$

Where the parameters k_{11} and k_{22} are related with the parameters k_1 and k_2, and

$$B = -2^{-4}$$

$$A = -[2^4 + 2]$$

The compensation filters (5.12) and (5.13) have many favourable characteristics such as parameters B and A equal in both compensators, and do not depend on M and M_1, filters are multiplier less, and can be moved to a lower rate but the number of cascade filters k_{11} and k_{22} depend on k_1 and k_2. The two-stage structure with the compensators is shown in figure below
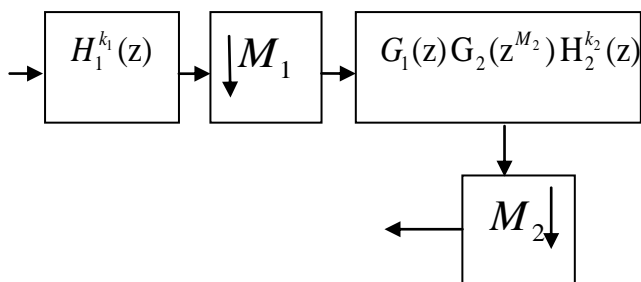

Fig 5.2:- Compensated two-stage structure

For proposed structure they apply the sharpening technique to the cascaded filters $H_2^{k_2}$ and the compensator $G_2(z^{M_2})$, at the second stage.

$$H_{2c}(z) = H_2^{k_2}(z)G_2(z^{M_2}) \qquad (5.14)$$

In the sharpening technique they used the simplest sharpening polynomial in the form of $2H - H^2$

$$Sh\{H_{2c}(z)\} = 2\,H_{2c}(z) - H_{2c}^2(z)$$

$$= H_{2c}(z)[2\,z^{-\tau} - H_{2c}(z)] \qquad (5.15)$$

Where, $Sh\{x\}$ means sharpening of x. The delay τ is introduced to keep the linear phase of the filter. The overall proposed transfer function is

$$H_p(z) = H_1^{k_1}(z)\,G_1(z^{M_1})\,Sh\{H_2^{k_2}(z^{M_1})\,G_2(z^M)\} \qquad (5.16)$$

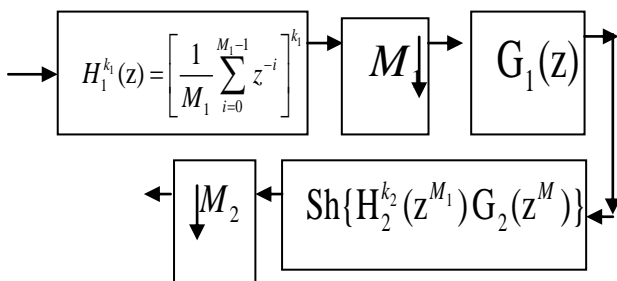The corresponding structure is shown in figure below

Fig 5.3:- Proposed Filter

This is the two-stage compensated sharpened COMB based decimator. The important features of the proposed filter are multiplier less structure and the design parameters are practically do

not depend on the values of M_1 and M_2. In the proposed design of the 2 stage COMB based decimation filter wideband pass band droop is very low and stop band attenuation is high of the overall filter.

5.2 Discussion:-

In paper 1 **Makarov and Odda** give the designing of second order COMB filter using delta modulation which gives the realization low cost, simplicity and an efficient result in real time processing. After that the suggested structure has been simulated through an appropriate computer program to achieve the desired frequency response. This structure is based on DM as analog to digital converter (ADC) for continuous input signal. In paper 2 **J. L. Rasmussen** gives the new method of designing a FIR COMB filter. They present a new formulation of the FFT COMB filter that eliminates all limitation of the FFT method and provides advantages over the IIR or FIR COMB filter. They design a COMB filter using FFT and IFFT matrix multiplication process. Due to this technique computational complexity is reduced and there is no restriction that a radix 2 value for the size of the FFT must be used but there is some restriction. N/M must be an integer, there is still requirement that all N samples must be collected before the output can be obtained and the time involved might be too large for some Time Division Multiple Access or network systems. In paper 3 **Gordana Jovanovic Dolecek** gives the designing of cascade comb filter. Using polyphase technique, compensation filter and sharpening technique they design a cascade comb filter. Due to this technique there are some advantages. We need amultiplier less structure and the design parameters are practically do not depend on the values of M_1 and M_2 but there is a one disadvantage that is number of cascade filters k_{11} and k_{22} depend on k_1 and k_2.

5.3 Comparison Table:-

Paper	Approach	Strength	Weakness
Paper 1	simulate second order COMB filter by using delta modulation (DM) technique	low cost, simplicity and an efficient result in real time processing	
Paper 2	Using FFT and IFFT matrix	Computational complexity is reduced	N/M must be an integer,

	multiplication process to design a COMB filter	and there is no restriction that a radix 2 value for the size of the FFT must be used.	there is still requirement that all N samples must be collected before the output can be obtained and the time involved might be too large for some Time Division Multiple Access or network systems.
Paper 3	Using polyphase technique, compensation filter and sharpening technique to design a cascade comb filter	Multiplier less structure and the design parameters are practically do not depend on the values of M_1 and M_2.	Number of cascade filters k_{11} and k_{22} depend on k_1 and k_2.

Table 5.1:- Comparison table of COMB filter desing

VI. Bessel Filter

Bessel filters [1] are a class of all-pole filters that are characterized by the system function

$$H(s) = \frac{1}{B_N(s)} \qquad (6.1)$$

Where $B_N(s)$ is the Nth-order Bessel polynomial. These polynomials can be expressed in the form

$$B_N(s) = \sum_{k=0}^{N} a_k s^k \qquad (6.2)$$

Where the coefficients { a_k } are given as

$$a_k = \frac{(2N-k)!}{2^{N-k} k!(N-k)!} \qquad k=0,1,.....,N \qquad (6.3)$$

Alternatively, the Bessel polynomial may be generated recursively from the relation

$$B_N(s) = (2N-1)B_{N-1}(s) + s^2 B_{N-2}(s) \qquad (6.4)$$

With $B_0(s) = 1$ and $B_1(s) = s+1$ as initial conditions.

An important characteristic of Bessel filters is the linear-phase response over the pass band of the filter. Bessel filter has a larger transition bandwidth, but its phase is linear within the pass band.

6.1 Designing Method
Paper 1

Johnson et al present [16] the new designing method of Bessel filter. They used the continued fraction to obtain the transfer function of Bessel filter is modified by adding a parameter a > -1 to the odd integers in the expansion. The resulting transfer function is realizable and has a time delay at w=0 which is independent of the order of the filter, and which may be varied from 0 to ∞ by changing α. So first we will discuss about the modified transfer function and then discuss about the time delay.

The transfer function of generalized Bessel filter is

$$H_n^a(s) = \frac{B_n^a(0)}{B_n^a(s)}, \quad a > -1 \qquad (6.5)$$

Where

$$B_n^a(s) = M_n^a(s) + N_n^a(s)$$

$M_n^a(s)$ and $N_n^a(s)$ are the even and odd part obtained from the continued fraction expansion

$$F_n^a(s) = \frac{M_n^a(s)}{N_n^a(s)} = \frac{a+1}{s} + \cfrac{1}{\cfrac{a+3}{s} + \cfrac{1}{\ddots + \cfrac{1}{\cfrac{a+2n-1}{s}}}}$$
$$(6.6)$$

When a=0, $B_n(s)$ is obtained i.e. $B_n(s) = B_n^0(s)$. $B_n^a(s)$ is Hurwitz if and only if a≥-1.

To obtain the expressions for B_n^a, from equation (6.6) note that

$$F_n^a(s) = \frac{a+1}{s} + \frac{1}{F_{n-1}^{a+2}(s)}$$

From which we obtain

$$\frac{M_n^a}{N_n^a} = \frac{(a+1)M_{n-1}^{a+2} + sN_{n-1}^{a+2}}{sM_{n-1}^{a+2}}$$

or

$$M_n^a(s) = (a+1)M_{n-1}^{a+2}(s) + sN_{n-1}^{a+2}(s)$$
$$N_n^a(s) = sM_{n-1}^{a+2}(s)$$

Therefore we may write

$$B_n^a(s) = sB_{n-1}^{a+2}(s) + (a-1)M_{n-1}^{a+2}(s) \qquad (6.7)$$

Or, since M_{n-1}^{a+2} is the even part of B_{n-1}^{a+2}, we have

$$M_{n-1}^{a+2}(s) = \frac{1}{2}\left[B_{n-1}^{a+2}(s) + B_{n-1}^{a+2}(-s) \right]$$

And therefore

$$B_n^a(s) = \frac{(2s+a+1)}{2}B_{n-1}^{a+2}(s) + \frac{(a+1)}{2}B_{n-1}^{a+2}(-s)$$

(6.8)

From equation (6.8)

$$B_n^a(s) = \sum_{k=0}^{n} A_k s^k, \qquad n=1,2,3.....$$ (6.9)

A list of the A_k for various value of n is given in Table 6.1, where they define

$$\alpha_i = a+i, \quad i = 1,2,3...$$ (6.10)

N	A_n	A_{n-1}	A_{n-2}	A_{n-3}	A_{n-4}
1	1	α_1			
2	1	α_3	$\alpha_1\alpha_3$		
3	1	$2\alpha_3$	$\alpha_3\alpha_5$	$\alpha_1\alpha_3\alpha_5$	
4	1	$2\alpha_5$	$3\alpha_3\alpha_5$	$\alpha_3\alpha_5\alpha_7$	$\alpha_1\alpha_3\alpha_5\alpha_7$
5	1	$3\alpha_5$	$3\alpha_5\alpha_7$	$4\alpha_3\alpha_5\alpha_7$	$\alpha_3\alpha_5\alpha_7\alpha_9$

Table 6.1:- Coefficient of B_n^a

The modified Bessel filter having transfer function (6.5) does not have maximally flat time delay, except in the case a=0 of the Bessel filter. However, by keeping a near 0, one may retain approximately the desirable phase properties of the Bessel filter while obtaining a variety of amplitude response.

By varies 'a' from 0 to in one direction or the other, observe the time delay of Bessel filter. Let us consider the time delay $T_n^a(w)$ of the modified Bessel filter at w=0. So,

$$T_n^a(w) = -\frac{d\phi(w)}{dw} = -\text{Re}\left(\frac{H'(jw)}{H(jw)} \right)$$

Where $H'(jw)$ represent differentiation with respect to (jw) and since both $H'(jw)$ and $H(jw)$ are real at w=0, then

$$T_n^a(0) = -\frac{H'(0)}{H(0)} = -H'(0)$$

Because $H_n^a(0) = 1$. Therefore,

$$T_n^a(0) = -\frac{d}{ds}H_n^a(s)|_{s=0}$$

From equations (6.5) and (6.9) is given by

$$T_n^a(0) = \frac{A_1}{A_0}$$ (6.10)

By equation (6.7) and (6.9), A_1 is equal to $B_{n-1}^{a+2}(0)$. But from (6.8), $B_{n-1}^{a+2}(0)$ is equal to $B_n^a(0) / (a+1)$, which is turn is $A_0 / (a+1)$. Substituting this expression for A_1 into equation (6.10)

$$T_n^a(0) = \frac{1}{a+1}$$ (6.11)

Thus the time delay at w=0 is independent of n, as in the case of Bessel filter. Also they note that if -1 < a < 0, then 1 < $T_n^a(0)$ < ∞, and if a > 0 then 0 < $T_n^a(0)$ < 1. The dividing line is the case of Bessel filter a = 0, in which case $T_n^a(0) = 1$.

The modified Bessel filter transfer function is a realizable function which reduces to that of the Bessel function when the parameter 'a' is zero. The time delay of the modified Bessel filter at w=0 is independent of the order of the filter and can be varied from 0 to ∞ by changing the parameter 'a'.

Paper 2

Another method of designing a Bessel filter is given by **Johnson et al** [17]**.** They present an extension of the Bessel filter where transfer function is a rational function with finite zeros. They combine the constant magnitude response of the all-pass filter with the linear phase response of the Bessel filter. This paper presents an extension to the Bessel filter based on interesting approximation suggested be Budak [18].

Budak's approximation is obtained by considering the ideal normalized transfer function e^{-s} in the form

$$e^{-s} = \frac{e^{-ks}}{e^{-(k-1)s}}, \quad k > 0$$ (6.12)

When the Bessel filter transfer function is used to independently approximate e^{-ks} and $e^{-(k-1)s}$, the Bessel rational filter approximant results, given by

$$e^{-s} \cong H_{mn}^k(s) = \frac{B_n(0)B_m[(k-1)s]}{B_m(0)B_n(ks)}, \quad m,n = 1,2,...,m \le n$$

(6.13)

When k=1 equation (6.13) becomes equal to equation (6.1). For $0 < k < 1$, equation (6.13) becomes nonminimum phase function and for $k \geq 1$, it is a minimum phase function.

Now, they obtained the time delay from that of the denormalized Bessel filter

$$\frac{d}{dw} \arg B_n(j\lambda w) = \lambda[1 - f_n(\lambda w)]$$

Where

$$f_n(w) = \frac{2}{\pi w[j^2_{n+\frac{1}{2}}(w) + j^2_{-n-\frac{1}{2}}(w)]}$$

And $j_{\pm(n+\frac{1}{2})}$ are the spherical Bessel function. For the normalized Bessel filter, the time delay is given by

$$T_n(w) = 1 - f_n(w) \qquad (6.14)$$

And for the normalized Bessel rational filter, the time delay is given by

$$T^k_{mn}(w) = 1 - kf_n(kw) - (1-k)f_m[(1-k)w] \qquad (6.15)$$

From (6.15) and the MacLaurin series for $f_n(u)$, given by

$$f_n(u) = \left[\frac{2^n n!}{(2n)!}\right]^2 \left[u^{2n} - \frac{u^{2n+2}}{2n-1} + \frac{2(n-2)u^{2n+4}}{(2n-1)^2(2n-3)} - ...\right]$$

They notice that the Bessel rational filter has maximally flat time delay in the case where a) k=1 (normalized Bessel Filter), b) m=0 (denormalized Bessel Filter), and c) m=n. There is an another important case of (6.13) where m=n and k=0.5, which is an all-pass filter with maximally flat time delay given by

$$T^{\frac{1}{2}}_{nn}(w) = 1 - f_n\left(\frac{w}{2}\right) \qquad (6.16)$$

From (6.14) it is note that the all-pass time delay is related to the Bessel filter by $T^{\frac{1}{2}}_{nn}(w) = T_n\left(\frac{w}{2}\right)$.

Therefore, the time delay of the all-pass filter is flat for approximately twice the frequency range as that of the Bessel filter of same order.

From the equation (6.2) Bessel polynomial can be defined by

$$B_n(s) = \sum_{i=0}^{n} \frac{(2n-i)! s^i}{2^{n-i}(n-i)! i!} \qquad (6.17)$$

By using (6.17) it is interesting to note that

$$B_n\left(\frac{s}{2}\right) = \frac{(2n)!}{2^n n!} P_n(s)$$

Where

$$P_n(s) = \sum_{i=0}^{n} \frac{n!(2n-i)! s^i}{(2n)!(n-i)! i!}$$

Thus for the special case, the Bessel rational filter can be written as

$$H^{\frac{1}{2}}_{nn}(s) = \frac{P_n(-s)}{P_n(s)}$$

Which is the (n,n) Pade [19] approximate of the function e^{-s}.

So, in this paper they used to Budak's and Pade approximate to design a transfer function of Bessel rational filter. They also combine the constant magnitude response of the all-pass filter with the linear phase response of the Bessel filter. Due to this they note that the time delay of the all-pass filter is flat for approximately twice the frequency range as that of the Bessel filter of same order.

Paper 3

Another method of designing a Bessel filter is given by **Susan and Jayalalitha** [20]. In this paper, they simulate the passive component L using Generalized Impedance Converter (GIC). This simulated inductor is applied for the realization of Bessel filter. The circuit simulation is done using PSPICE. The use of passive component such as inductor in analog circuit is very difficult at low frequencies because the size of the inductor becomes very large. Due to this reason they simulate the inductor and then this inductor used for realization of Bessel filter.

For simulating the inductor they used the GIC. They obtained the simulated inductor circuit from the GIC which consists of the active component the operational amplifier along with the capacitor. The GIC invented by Antoniou is given in figure (6.1).
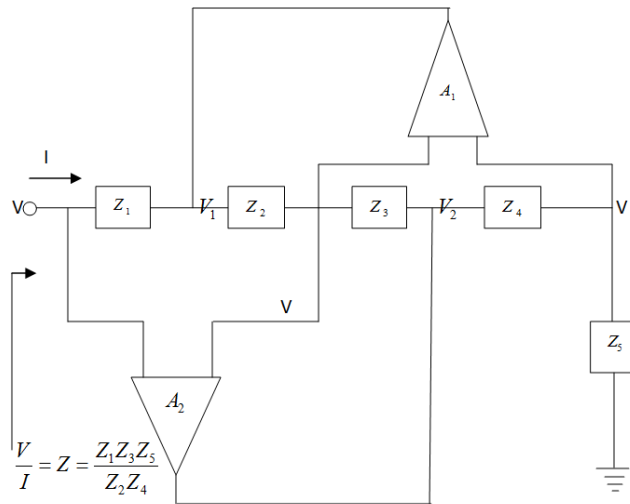
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Fig 6.1:- Antoniou

Circuit

The input impedance with respect to ground is

$$Z = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} \qquad (6.18)$$

Where

$$Z_1 = R_1, Z_2 = X_{C2}, Z_3 = R_3, Z_4 = R_4, Z_5 = R_5$$

then,

$$Z = \frac{sR_1 R_3 R_5 C_2}{R_4} \qquad (6.19)$$

This is equivalent to an inductor. The value of L is

$$L = \frac{R_1 R_3 R_5 C_2}{R_4}$$

If $R_1 = R_3 = R_5 = R_4 = R$ and $C_2 = C$ then

$$L = CR^2 \qquad (6.20)$$

From this procedure they obtained the value of simulated inductor. Now this value of simulated inductor they used to design of Bessel filter.

For simulation of Bessel filter they replaced the inductor L in the low pass filter by the simulated inductor as shown in fig (6.2) and the frequency response obtained using PSPICE is given in fig (6.3).
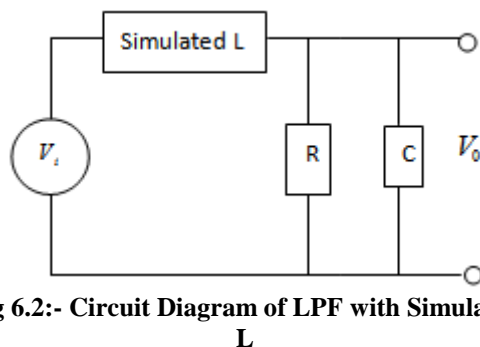

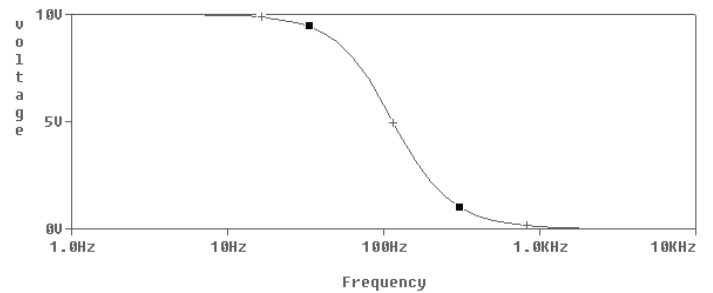**Fig 6.2:- Circuit Diagram of LPF with Simulated L**

Fig 6.3:- Frequency Response of Low Pass Filter

Similarly to design a Bessel filter they replaced the inductor L in the basic circuit of the high pass and all-pass filters by the simulated inductor as shown in fig (6.5) and (6.6). The frequency response obtained using PSPICE is given in fig (6.7) and (6.8).
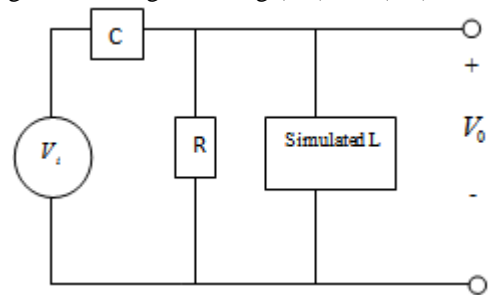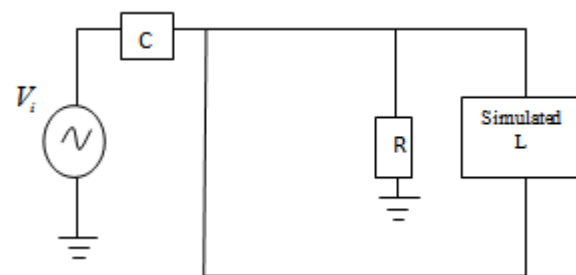

Fig 6.5:- High Pass Filter using Simulated L

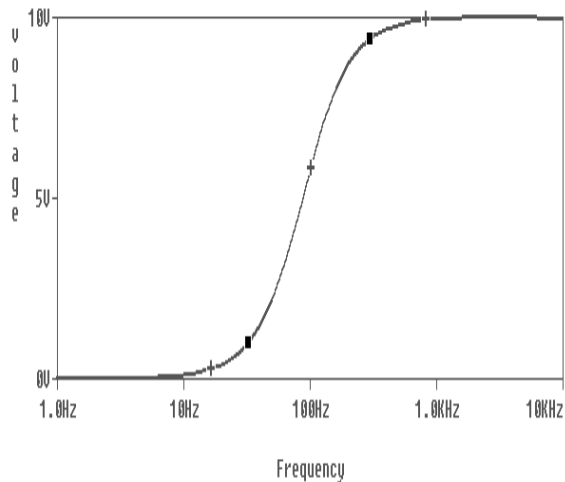**Fig 6.6:- All-Pass Filter using Simulated**

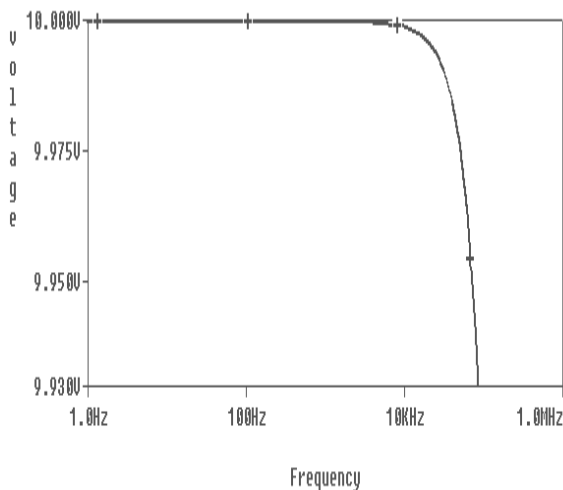**Fig 6.7:- Frequency Response of High Pass Filter**

Fig 6.8:- Frequency Response of All-Pass Filter

In this paper first they obtained the simulated inductor using GIC then design of Bessel filter from the low pass, high pass and all-pass filter with the help of simulated inductor.

Paper 4

Another method for designing Bessel filter is given by **Chengyun et al** [21]**.** In this paper a method for the design of digital Bessel filter is presented. It is based on MATLAB function besselap and zp2sos to obtain the Bessel polynomials. They obtain the analog filter transform function $H_a(\text{s})$ by search N-th order normalized Bessel polynomials. Then, transform $H_a(\text{s})$ into H(z) using the bilinear transformation and obtain the general coefficients function of digital Bessel filters from H(z). MATLAB provides a function called bilinear to implement this mapping. This mapping is the best

transformation method named as bilinear transformation. To get the Bessel polynomials they used the following MATLAB function
[z,p,k]=besselap(N)   % N-th order normalized prototype Bessel analog low pass filter
[sos,g]=zp2sos(z,p,k)   %Zero-pole-gain to second order section model conversion
Using this MATLAB code they obtained polynomials factors as shown in table 6.1

Filter Order	$D'(\text{p}) = B_1^{'}(\text{p})\, B_2^{'}(\text{p})\, B_3^{'}(\text{p})\, B_4^{'}(\text{p})$ **Polynomials Factors**
1	(p+1)
2	(p^2 +1.7321p +1)
3	(p^2 + 1.4913p + 1.0620)(p + 0.9416)
4	(p^2 + 1.3144p + 1.1211)(p^2 + 1.8095p + 0.8920)
5	(p^2 + 1.1812p + 1.1718)(p^2 + 1.7031p + 0.9211)(p + 0.9264)

Table 6.1:- N-order normalized polynomials of Bessel

In designing of Bessel digital filter, first they design the Bessel digital low pass filter. For this they obtained the N-th order normalized Bessel analog low-pass filter transfer function from table 6.1

$$G(\text{p}) = \frac{1}{D_N^{'}(\text{p})} \qquad (6.21)$$

Then they obtained the N-th order Bessel order analog low-pass filter transform function by removing normalization of equation (6.21)

$$H_a(\text{s}) = G(\text{p})\big|_{p=\,{}^{s}\!/_{\Omega_c}} \qquad (6.22)$$

Where Ω_c is pass band cut-off frequency in rad/sec. Now, they transform $H_a(\text{s})$ into H(z) using the bilinear transformation. They obtained

$$H(\text{z}) = H_a(\text{s})\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \prod_{k=1}^{M} \frac{b_{0k} + b_{1k}z^{-1} + b_{2k}z^{-2}}{a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}} \qquad (6.23)$$

Any multi-order Bessel filter can be designed by using normalized Bessel polynomials presented in Table 6.1. The general form of the first order polynomial is:
$$D'(\text{p}) = \text{p} + \text{d} \qquad (6.24)$$

From equations (6.21), (6.22) and (6.23) they obtain relevant digital filter coefficients B, A (where $B = [b0, b1, b2], \quad A = [a0, a1, a2]$):

$$B = \left[\frac{\Omega_c}{1+d\Omega_c}, \frac{\Omega_c}{1+d\Omega_c}, 0 \right] \qquad (6.25)$$

$$A = \left[1, \frac{d\Omega_c - 1}{1+d\Omega_c}, 0 \right] \qquad (6.26)$$

Now, they consider the general form of second order polynomial is;

$$D'(\mathrm{p}) = \mathrm{p}^2 + kp + c \qquad (6.27)$$

Again from equations (6.21), (6.22) and (6.23) they obtain relevant digital filter coefficients B, A:

$$B = \left[\frac{\Omega_c^2}{1+k\Omega_c+c\Omega_c^2}, \frac{2\Omega_c^2}{1+k\Omega_c+c\Omega_c^2}, \frac{\Omega_c^2}{1+k\Omega_c+c\Omega_c^2} \right]$$
$$(6.28)$$

$$A = \left[1, \frac{2c\Omega_c^2 - 2}{1+k\Omega_c+c\Omega_c^2}, \frac{1-k\Omega_c+c\Omega_c^2}{1+k\Omega_c+c\Omega_c^2} \right] \quad (6.29)$$

From this method they obtain the general polynomials coefficients of Bessel filter.

After designing of Bessel low-pass digital filter, they design the Bessel high-pass filter by similar way. The difference is removing normalization. For removing normalization they used the following equation:

$$H_a(\mathrm{s}) = G(\mathrm{p})\big|_{p=\Omega_c/s} \qquad (6.30)$$

They obtained relevant digital filter coefficients B, A of order 2 (where $B = [b0, b1, b2], \quad A = [a0, a1, a2]$)

$$B = \left[\frac{1}{c+k\Omega_c+\Omega_c^2}, \frac{-2}{c+k\Omega_c+\Omega_c^2}, \frac{1}{c+k\Omega_c+\Omega_c^2} \right]$$
$$(6.31)$$

$$A = \left[1, \frac{2\Omega_c^2 - 2c}{c+k\Omega_c+\Omega_c^2}, \frac{c-k\Omega_c+\Omega_c^2}{c+k\Omega_c+\Omega_c^2} \right]$$
$$(6.32)$$

And the relevant digital filter coefficients B, A of order 1 is:

$$B = \left[\frac{1}{d+\Omega_c}, \frac{-1}{d+\Omega_c}, 0 \right]$$

$$A = \left[1, \frac{\Omega_c - d}{\Omega_c + d}, 0 \right]$$

To ensure the methods mentioned above correct they validate them by MATLAB program and determine the frequency response for 5th order Bessel digital filter as shown in figure below.
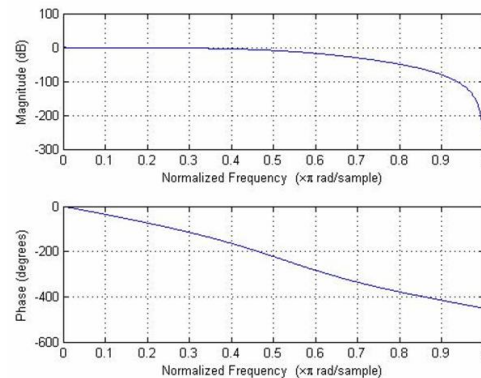

Fig 6.9:- Bessel digital low-pass filter frequency response
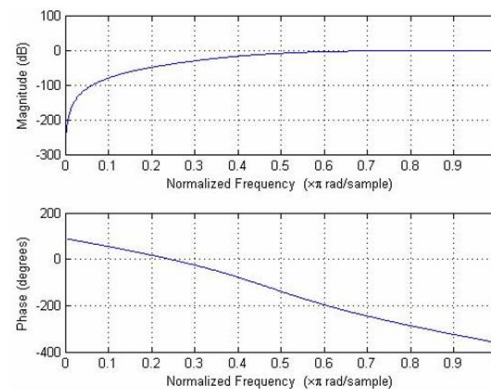

Fig 6.10:- Bessel digital low-pass filter frequency response

The results of figure (6.9) and (6.10) show that the designed filter meet the requirements.

6.2 Discussion:-

In the paper 1, **Johnson et al present** the new designing method of Bessel filter. They used the continued fraction to obtain the transfer function of Bessel filter .The resulting transfer function is realizable and has a time delay at w=0 which is independent of the order of the filter, and which may be varied from 0 to ∞ by changing α. But adding a parameter, in a Bessel filter transfer function, a > -1 to the only for odd integers in the expansion not for even integer. In paper 2, **Johnson et al** present an extension of the Bessel filter where transfer function is a rational function with finite zeros. They combine the constant magnitude response of the all-pass filter with the linear phase response of the Bessel filter. In this paper they used the Budak and Pade approximation to design Bessel filter. In paper 3, **Susan and Jayalalitha** simulate the passive component L using GIC. This simulated inductor is applied for the realization of Bessel filter. The circuit simulation is done using PSPICE. After that this

simulated L use in low-pass, high-pass and all-pass filter and design Bessel filter and get a maximally flat response being equivalent to a time delay and linear phase response. This simulated L can be used for the design low frequency analog circuits which make use of the high value of unrealized L. In paper 4, designing of Bessel filter is given by **Chengyun et al.** In this paper a method for the design of digital Bessel filter is presented. It is based on MATLAB function besselap and zp2sos to obtain the Bessel polynomials. Using the bilinear transformation they obtain the general coefficients function of digital Bessel filters from H(z) and verified by MATLAB. Linear phase along with good magnitude flatness is obtained and method of determine the general coefficients function of digital Bessel filter can be deduced.

6.3 Comparison Table:-

Paper	Approach	Strength	Weakness
Paper 1	Using the continued fraction to obtain the transfer function of Bessel filter.	At w=o time delay is independent of order of Bessel filter and the modified Bessel filter transfer function is a realizable.	Adding a parameter, in a Bessel filter transfer function, a > -1 to the only for odd integers in the expansion.
Paper 2	Combine the constant magnitude response of the all-pass filter with the linear phase response of the Bessel filter	Time delay of the all-pass filter is flat for approximately twice the frequency range as that of the Bessel filter of same order.	Step response has a nonzero initial value. This could be undesirable for some applications.
Paper 3	Obtained the simulated inductor circuit from the GIC which is use for realization of Bessel filter.	Due to the simulated inductor the use of passive component such as inductor in analog circuit is very easy at low frequencies.	

Paper	Approach	Strength	Weakness
Paper 4	Using the bilinear transformation obtain the general coefficients function of digital Bessel filters from H(z) and verified by MATLAB.	Linear phase along with good magnitude flatness is obtained and method of determine the general coefficients function of digital Bessel filter can be deduced.	

Table 6.2:- Comparison Table of Bessel Filter

VII. Conclusion

Filter can be designed in many ways. In this paper we discussed the Butterworth filter, Chebyshev filter, Notch filter, Comb filter and Bessel filter and gave the designing of filters on the basis of some standard paper which is based on filter design. In **Butterworth filter**, we discussed the four papers which is based upon the Sallen-key topology ,double precision method and take a nth order pole from (n+2)th order filter to improve the filter methods. In **Chebyshev filter,** we discussed the three papers which is based upon shifting the frequencies nearest to origin and not change in the cut-off frequency, use magnitude response of low pass filter and convert into Chebyshev filter magnitude response and Convert 1D Chebyshev polynomial into 2D polynomial. In **Notch filter,** we discussed the four papers which is based upon simple algebraic method, largen method, MATLAB and pole-position method. In **Comb filter**, we discussed the three papers which is based upon delta modulation, matrix multiplication and polyphase technique. In **Bessel filter,** we discussed four papers which is based upon continue fraction, combine the all pass filter, GIC and MATLAB methods. After discussing the methods of each filter we discussed the strength and weakness of each method in comparison table and give the review of filters.

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